A General Transport Rule for Variable Mass Dynamics

STUART B. BELKNAP*
Aerospace Corporation, El Segundo, Calif.

Theme

MASS increments and associated dynamic properties are transported across the boundary of a variable mass system, and this complicates the derivation and interpretation of equations of motion. Of course, various formulations of variable mass dynamics exist in the literature and in standard texts. However, ad hoc methods are generally applied to the transport problem, results differ greatly in form and complexity, and the origin and significance of variable mass peculiar terms is often not clear.

In this paper, a general transport rule is applied as a contribution toward unity and clarity in variable mass dynamics. Also, results are obtained in forms similar to familiar constant mass relations. These forms are convenient for analysis or simulation and for evaluating the relative significance of variable mass peculiar terms.

Content

Background: Figure 1 illustrates a general system in which Δm is an increment of mass m enclosed by arbitrary control surface S of volume V. Often, S corresponds to the exterior of a vehicle. Frame A is an inertial reference system with origin at point A, frame B is an arbitrary reference system with origin at point B, and frame C is parallel to B centered at instantaneous center of mass C. Point B may or may not be fixed relative to the rigid structure of a vehicle. Various position vectors are defined in Fig. 1. Among these, $\vec{r}_C = r_{C\alpha}\hat{e}_\alpha = (1/m)\int_V \vec{r}\, dm$ locates the center of mass relative to point B. Dyadic $\mathbf{I}_B = I_{B\beta\alpha}\hat{e}_\beta\hat{e}_\alpha = \int_V (\vec{r}\cdot\vec{r}\mathbf{E} - \vec{r}\vec{r})\, dm$ specifies the inertia of the system about B in terms of the unit dyadic $\mathbf{E}$. In these relations, the convention is to sum on repeated dummy indices.

A dot is employed to indicate a time derivative. A superscript, as in $\dot{\vec{r}}^A = d\vec{r}/dt^A$ and in $\dot{\mathbf{I}}_B^B = d\mathbf{I}_B/dt^B$, is used to indicate the reference frame of a vector time derivative. If $\vec{w}$ is an arbitrary vector and $\vec{\omega}$ is the angular velocity of frame B or parallel frame C relative to A, then

$$\dot{\vec{w}}^A = \dot{\vec{w}}^B + \vec{\omega} \times \vec{w} \tag{1}$$

$$\ddot{\vec{w}}^A = \ddot{\vec{w}}^B + 2\vec{\omega} \times \dot{\vec{w}}^B + \vec{\omega} \times (\vec{\omega} \times \vec{w}) + \dot{\vec{\omega}}^B \times \vec{w} \tag{2}$$

Translational and rotational differential equations are obtained by integrating Newton's laws of motion over the mass increments of a system. Results are as follows, in which $\vec{F}$ is net external force, $\vec{M}_B$ is net external moment about point B, and $\vec{v} = \dot{\vec{r}}^A$ is the inertial velocity of Δm

$$\int_V (d\vec{v}/dt^A)\, dm = \vec{F} \tag{3}$$

$$\int_V (d/dt^A)(\vec{r} \times \dot{\vec{r}}^A)\, dm = -\vec{r}_C \times m\ddot{\vec{R}}_B^A + \vec{M}_B \tag{4}$$

These relations are valid for constant or for variable mass systems but are not convenient for analysis or simulation due

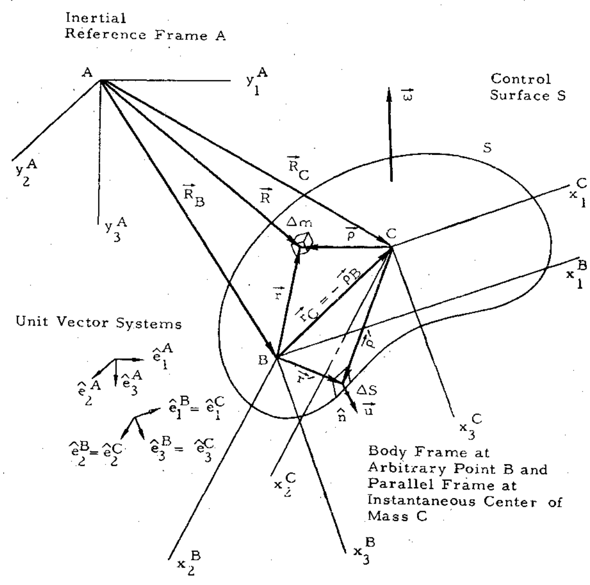

Fig. 1 General system.

to the location of the time derivatives interior to the volume integrals.

General Transport Rule: Let ψ represent any set of tensor components associated with Δm. For example, if $\psi = 1$, then $\psi\Delta m = \Delta m$ is an increment of mass. If $\psi = \vec{v}$, then $\psi\Delta m = \vec{v}\Delta m$ is a vector representing the three components of an increment of linear momentum. The general transport rule, in derivative form, is expressed in Eq. (5). Here, $\dot{m}_s$ is the mass per area per time transported across ΔS, positive for efflux

$$\int_V (d\psi/dt)\, dm = (d/dt)\int_V \psi dm + \int_S \psi \dot{m}_s\, dS \tag{5}$$

This relation provides a rule for extracting the time derivatives in Eqs. (3) and (4) and for performing several other useful manipulations related to the transport of dynamic properties as detailed below.

Translational and rotational motion equations: To obtain translational motion Eq. (6), apply the transport rule with $\psi = \vec{v}$ to Eq. (3) and use the continuity relation obtained by substituting $\psi = 1$ in the transport rule. Substitute $\vec{v} = \vec{v}_B + \dot{\vec{r}}^A$, let $\vec{u}$ represent the velocity of Δm relative to ΔS at the surface, and let $\dot{\vec{r}}'^A$ be the inertial velocity of ΔS relative to B

$$m\dot{\vec{v}}_B^A = \vec{F} - (d/dt^A)\int_V \dot{\vec{r}}^A\, dm - \int_S (\dot{\vec{r}}'^A + \vec{u})\dot{m}_s\, dS \tag{6}$$

The transport rule with $\psi = \vec{r}$ is used to derive the identity expressed in Eq. (7)

$$(d/dt^A)\int_V \dot{\vec{r}}^A\, dm = m\ddot{\vec{r}}_C^A + \ddot{m}\vec{r}_C + 2\dot{m}\dot{\vec{r}}_C^A + (d/dt^A)\int_S \vec{r}'\dot{m}_s\, dS \tag{7}$$

Substitute this relation in Eq. (6) with $\dot{\vec{v}}_C^A = \dot{\vec{v}}_B^A + \ddot{\vec{r}}_C^A$ to produce an alternate form for the translational motion relations

Received August 9, 1971; synoptic received March 2, 1972. Full paper is available from National Technical Information Service, Springfield, Va., 22151, as N72-21570 at the standard price (available upon request).

Index category: Uncontrolled Rocket and Missile Dynamics.

* Member of the Technical Staff, Performance Analysis Department.

$$m\dot{\vec{v}}_C^A = \vec{F} - \dot{m}\dot{\vec{r}}_C - 2\dot{m}\dot{\vec{r}}_C^A - (d/dt^A)\int_S \vec{r}'\dot{m}_s\,dS - \int_S (\dot{\vec{r}}'^A + \vec{u})\dot{m}_s\,dS \qquad (8)$$

The origin and significance of the $\ddot{m}\vec{r}_C$ and $2\dot{m}\dot{\vec{r}}_C^A$ terms in Eq. (8) is often questioned. The transport rule, as expressed in Eq. (7), shows that these terms involve nothing more than an alternate representation of the rate of change of internal linear momentum.

To obtain rotational motion equation (9), apply the transport rule with $\psi = \vec{r} \times \dot{\vec{r}}^A$ to Eq. (4). Use $\dot{\vec{r}}^A = \dot{\vec{r}}^B + \vec{\omega} \times \vec{r}$, identify the inertia dyadic $\mathbf{I}_B$, and let $\vec{H}_B' = \int_V \vec{r} \times \dot{\vec{r}}^B\,dm$ represent system angular momentum relative to frame B

$$\mathbf{I}_B \cdot \dot{\vec{\omega}}^B + \vec{\omega} \times (\mathbf{I}_B \cdot \vec{\omega}) + \dot{\mathbf{I}}_B^B \cdot \vec{\omega} + \dot{\vec{H}}_B'^A = -\vec{r}_C \times m\ddot{\vec{R}}_B^A + \vec{M}_B - \int_S \vec{r}' \times \vec{u}\dot{m}_s\,dS - \int_S \vec{r}' \times \dot{\vec{r}}'^B \dot{m}_s dS - \int_S \vec{r}' \times (\vec{\omega} \times \vec{r}')\dot{m}_s\,dS \qquad (9)$$

If center of mass C is used as the basic reference point in Eqs. (9) and (10), C replaces B, $\vec{\rho}$ replaces $\vec{r}$, and terms containing $\vec{\rho}_C$ are zero. In general, of course, point C moves relative to the rigid structure of a vehicle due to internal rearrangement or transport of mass across surface S.

Various definitions of jet damping appear in the literature. In this paper, a jet damping dyadic is defined according to $\mathbf{D}_B = \int_S (\vec{r}' \cdot \vec{r}'\mathbf{E} - \vec{r}'\vec{r}')\dot{m}_s\,dS$. Then, the jet damping moment $\vec{M}_B^{(D)} = -\int_S \vec{r}' \times (\vec{\omega} \times \vec{r}')\dot{m}_s\,dS = -\mathbf{D}_B \cdot \vec{\omega}$ represents the rate of transport of angular momentum across S due to rigid body rotation. In these terms, Eq. (9) is written

$$\mathbf{I}_B^B \cdot \dot{\vec{\omega}}^B + \vec{\omega} \times (\mathbf{I}_B \cdot \vec{\omega}) + (\dot{\mathbf{I}}_B^B + \mathbf{D}_B) \cdot \vec{\omega} + \dot{\vec{H}}_B'^A = -\vec{r}_C \times m\ddot{\vec{R}}_B^A + \vec{M}_B - \int_S \vec{r}' \times \vec{u}\dot{m}_s\,dS - \int_S \vec{r}' \times \dot{\vec{r}}'^B \dot{m}_s\,dS \qquad (10)$$

Here $\dot{\mathbf{I}}_B^B$ is due to internal rearrangement or transport of mass across S and may be positive or negative. Dyadic $\mathbf{D}_B$ is due to mass crossing S and is positive for efflux. The following fact is often not recognised. The part of $\dot{\mathbf{I}}_B^B$ due to transport is exactly, and in all cases, annulled by $\mathbf{D}_B$. To show this, apply the transport rule of Eq. (5) with $\psi = (\vec{r} \cdot \vec{r}\mathbf{E} - \vec{r}\vec{r})$. Obtain

$$\dot{\mathbf{I}}_B^B + \mathbf{D}_B = \int_V (d/dt^B)(\vec{r} \cdot \vec{r}\mathbf{E} - \vec{r}\vec{r})\,dm \qquad (11)$$

Equation (11) establishes a useful simplification for externally variable mass systems, defined as configurations for which internal relative motion of mass increments is negligible. For example, consider ablating systems and vehicles driven by small peripheral rockets. In this case, the $(\dot{\mathbf{I}}_B^B + \mathbf{D}_B) \cdot \vec{\omega}$ term in Eq. (10) is negligible.

Translational equations (6) and (8) and rotational equations (9) and (10) are convenient for the analysis or simulation of variable mass systems. Inertial time derivatives are used in (6) and (8) because translational equations are most conveniently integrated in inertial coordinates. Relative time derivatives are used in (9) and (10) because rotational equations are most conveniently integrated in body coordinates. In a general simulation, translational and rotational relations are integrated simultaneously along with kinematic differential equations and perhaps with additional motion equations derived from appropriate subsystem models. External force $\vec{F}$ and moment $\vec{M}_B$ include any force or moment due to pressure or inertial forces on surface S due to material exterior to S.

General rocket vehicle: Assume that all transport occurs through a system of n rocket nozzles characterized by: effective locations $\vec{r}_{e_i}$ and $\vec{\rho}_{e_i}$, and velocities $\dot{\vec{r}}_{e_i}^A$ and $\dot{\vec{\rho}}_{e_i}^A$, relative to points B and C. Let $\vec{u}_{e_i}$ represent the effective jet velocity at a nozzle and define m_i as the associated mass rate, negative for efflux.

Then, translational motion equation (8) assumes the following form:

$$m\dot{\vec{v}}_C^A = \vec{F} + \sum_i^n \vec{u}_{e_i}\dot{m}_i + \sum_i^n 2\dot{\vec{\rho}}_{e_i}^A m_i + \sum_i^n \vec{\rho}_{e_i}\ddot{m}_i \qquad (12)$$

Rotational motion relation (10) is written

$$\mathbf{I}_B \cdot \dot{\vec{\omega}}^B + \vec{\omega} \times (\mathbf{I}_B \cdot \vec{\omega}) + (\dot{\mathbf{I}}_B^B + \mathbf{D}_B) \cdot \vec{\omega} + \dot{\vec{H}}_B'^A = -\vec{r}_C \times m\ddot{\vec{R}}_B^A + \vec{M}_B + \sum_i^n \vec{r}_{e_i} \times \vec{u}_{e_i}\dot{m}_i + \sum_i^n \vec{r}_{e_i} \times \dot{\vec{r}}_{e_i}^B m_i - \sum_i^n \int_S (\vec{r}' - \vec{r}_{e_i}) \times \vec{u}\dot{m}_s\,dS - \sum_i^n \int_S (\vec{r}' - \vec{r}_{e_i}) \times \dot{\vec{r}}'^B \dot{m}_s\,dS \qquad (13)$$

In which

$$\mathbf{D}_B \cdot \vec{\omega} = -\sum_i^n \vec{r}_{e_i} \times (\vec{\omega} \times \vec{r}_{e_i})\dot{m}_i + \sum_i^n \int_S (\vec{r}' - \vec{r}_{e_i}) \times (\vec{\omega} \times \vec{r}')\dot{m}_s\,dS \qquad (14)$$

Certain symmetry is required for the integral terms to be zero. These terms may be non-zero if there is ordered swirling at a nozzle exit plane. Equations (12–14) provide a convenient basis for evaluating the significance of variable mass peculiar terms on the basis of relative magnitude, symmetry, or uniformity. The inertia rate dyadic $\dot{\mathbf{I}}_B^B$ and the jet damping dyadic $\mathbf{D}$ are not, in general, negligible. However, as explained previously, the dyadic sum $(\dot{\mathbf{I}}_B^B + \mathbf{D}_B)$ is negligible to the extent to which the vehicle can be modeled as an externally variable mass system. Finally, for a symmetrical rocket and steady flow, in the absence of internal moving parts, the relative angular momentum term $\dot{\vec{H}}_B'^A$ is usually negligible.